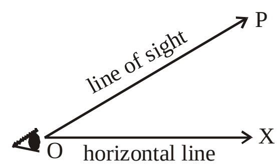
7 HEIGHT & DISTANCES (APPLICATION OF TRIGONOMETRY)

7.1 INTRODUCTION

In this chapter, we shall be applying trigonometric result's to discuss simple problem's based on height and distance. One of the object of trigonometry is to find the distance between two points or the height of tower, building and the height of difinite objects without actually measuring these distance or these height's. We begin by some definitions which will be user in this chapter.

7.2 LINE OF SIGHT

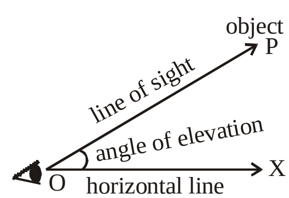
It is the line from the eyes of the observer to a point where the person is viewing.



In figure, the eye is at point O and the object is at P therefore OP is the line of sight.

7.2.1 Angle of Elevation

It is the angle formed by the line of sight with horizontal through the eyes, of observer, when the object is above the horizontal level.

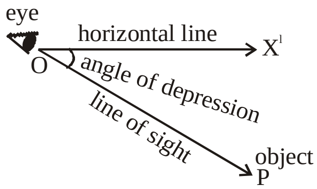


In figure the eye is at point O and the position of the object is 'P'. Therefore OP is the line of sight which make an angle XOP form horizontal line OX.

Hence the angle of elevation = $\angle XOP$.

7.2.2 Angle of Depression

It is the angle formed by the line of sight with the horizontal when the object is below the horizontal level.



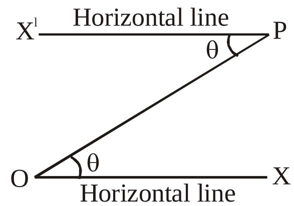
In figure the eye is at point O and the position of the object is P therefore OP is the line of sight which makes an angle X'OP form Horizonal line. Hence the angle XOP is the angle of depression = $\angle$X'OP.

Note: While solving the problem's of height and distance following point's must be noted.

(i) First of all read the question carefully and draw the figure.

(ii) In right traingle trigonometric ratio of known angles (sine, cosine, tangent etc.) are express in the term of knwon side.

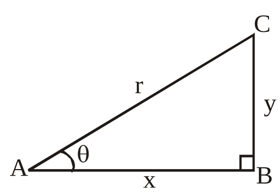
(iii) From given figure it is clear that the angle of elevation of 'O' with respect to 'P' is equal to the angle of depression of 'P' with respect to O'. I.e. the angle of elevation of one object is equal to the angle of depression of the other object with respect to the first object.



7.2.3 More important result

Let $\angle BAC = \theta$ be on acute angle of a right-angled Δ ABC.

We define the following ratios, known as trigonometric ratios for θ



(i) $\sin\theta = \dfrac{\text{perpendicular}}{\text{hypotenuse}} = \dfrac{y}{r}$

(ii) $\cos\theta = \dfrac{\text{base}}{\text{hypotenuse}} = \dfrac{x}{r}$

(iii) $\tan\theta = \dfrac{\text{prependicular}}{\text{base}} = \dfrac{y}{x}$

Reciprocal Relation

(i) $\text{cosec}\,\theta = \dfrac{1}{\sin\theta}$, (ii) $\sec\theta = \dfrac{1}{\cos\theta}$, (iii) $\cot\theta = \dfrac{1}{\tan\theta}$

Illustration 1

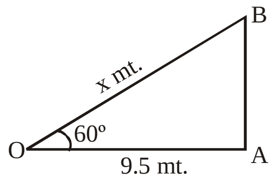
The angle of elevation of a ladder leaning against a wall is 60^o and the foot of the ladder is 9.5 mt. away form the wall. Find the length of the ladder.

Solution

Let OB be the ladder resulting against the wall AB, then $\angle AOB = 60^\circ$ and OA = 9.5 mt.

Let length of the ladder = OB = x mt.

In right angles triangle OAB.



$$\therefore \quad \frac{OB}{OA} = \sec 60^\circ$$

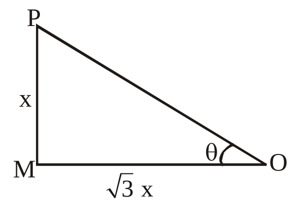
or $\frac{x}{9.5} = 2$ $\qquad \therefore$ x = (9.5 × 2) mt. = 19 mt.

Hence length of ladder is 19 mt.

Illustration 2

If the length of a shadow cost by a pole be $\sqrt{3}$ times the length of the pole, find the angle of elevation of the sun.

Solution



Let MP be the pole, then shadow

OM = $\sqrt{3}$ MP (given)

Let $\angle MOP = \theta$, the elevation of the sun

From right angled Δ OMP, we get

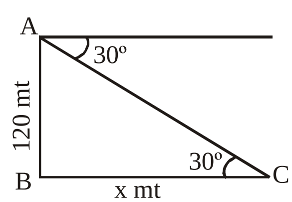
$$\tan\theta = \frac{MP}{OM} = \frac{x}{x\sqrt{3}} = \frac{1}{\sqrt{3}} = \tan 30^\circ$$

$\Rightarrow \quad \theta = 30^\circ$

Illustration 3

From the top of a tower 120 mt. high, the angle of depression of a car is 30º, find how far is the car from the tower.

Solution



Let AB be the tower and C be position of a car,

then AB = 120º and

let BC = x mt. then

$$\frac{BC}{AB} = \cot 30^\circ$$

$$\Rightarrow \quad \frac{x}{120} = \sqrt{3}$$

$\Rightarrow \quad x = 120\sqrt{3}$

$\Rightarrow \quad$ x = 207.840 mt.

Hence distance of the car from the top = 207.84 mt.

Illustration 4

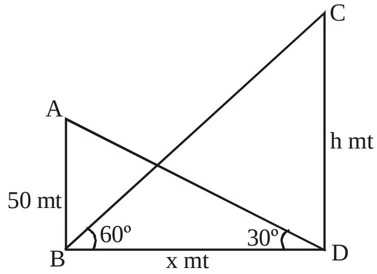
The angle of elevation of the top of a hill at the foot of a tower is 60º and the angle of elevation of the top of the tower from the foot of hill is 30º. If the tower is 50 mt. high, what is the height of the hill.

Solution

Let AB be the tower and CD be the hill

Let CD = h mt and BD = x mt.

in Δ ABD, we have



$$\frac{50}{x} = \tan 30^\circ$$

$\Rightarrow$ $x = 50\sqrt{3}$ mt. .....(i)

in Δ BCD, we have

$h/x = \tan 60^\circ$

$\Rightarrow$ $h = x \tan 60^\circ$ .....(ii)

Substitute value of x in (ii)

$h = 50\sqrt{3} \times \sqrt{3}$

$h = 150$ mt.

Height of hill is 150 mt.

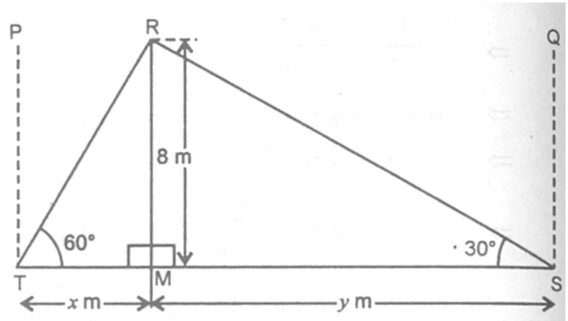
Solved Examples

Example 1

Ranjan is sitting at a height of 8 m on a tall tree on a small in the middle of a river. He observes two poles directly opposite each other on the two banks of the river and in line with the foot of the tree. If the angles of depression of the feet of the poles from the point at the which Ranjan is sitting on either side of the river side of the river are 60° and 30° respectively, find the width of the river.

Solution

Let Ranjan is sitting at on the tree RM which is 8 m high. PT and QS are the two poles on the opposite banks of the river.



Let the distance TM and MS be x and y respectively.

From right triangle TMR, $\tan 60° = \frac{RM}{x}$

$\Rightarrow \quad \sqrt{3} = \frac{8}{\sqrt{x}}$

$\Rightarrow \quad x = \frac{8}{\sqrt{3}}$ .... (i)

From right triangle SMR, $\tan 30° = \frac{8}{y}$

$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{8}{y}$

$\Rightarrow \quad y = 8\sqrt{3}$ .... (ii)

The width of the river isgiven by

$x + y = \frac{8}{\sqrt{3}} + 8\sqrt{3}$

$= \frac{8+24}{\sqrt{3}} = \frac{32}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$

$= \frac{32\sqrt{3}}{3} = \frac{32 \times 1.732}{3} = \frac{55.424}{3}$ = 18.475 metres. (Taking $\sqrt{3}$ = 1.732.)

$\therefore$ **The width of the river is 18.475 metres.**

Example 2

As observed from the top of a light house, 100m above sea level, the angle of depression of a ship, sailing directly towards it, changes from 30° to 45°. Determine the distance travelled by the ship during the period of observation.

Solution

Let A and B be the two positions of the ship. Let d be the distance travelled by the ship during the period of observation i.e. AB = d metres.

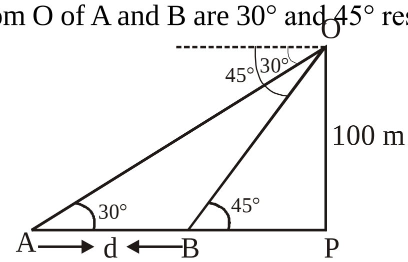
Let the observer be at O, the top of the light house PO.

It is given that PO = 100 m and the angles of depression from O of A and B are 30° and 45° respectively.

$\therefore \quad \angle OAP = 30°$ and $\angle OBP = 45°$,

In Δ OPB, we have

$\tan 45° = \frac{OP}{BP}$



$\Rightarrow 1 = \frac{100}{BP}$

$\Rightarrow$ BP = 100 m

In Δ OPA, we have

$\Rightarrow \tan 30° = \frac{\text{OP}}{\text{AP}}$

$\Rightarrow \frac{1}{\sqrt{3}} = \frac{100}{d + BP}$

$\Rightarrow d + BP = 100\sqrt{3}$

$\Rightarrow d + 100 = 100\sqrt{3}$

$\Rightarrow d = 100\sqrt{3} - 100$

$\Rightarrow d = 100\left(\sqrt{3} - 1\right) - 100(1.732 - 1) = 73.2\ m$

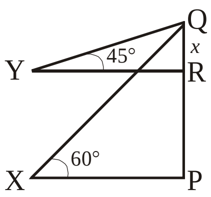
Hence, the distance travelled by the ship from A to B is 73.2 m.

Example 3

The angle of elevation of the top Q of a vertical tower PQ from a point X on the ground is 60°. At a point Y, 40m vertically above X, the angle of elevation is 45°. Find the height of the tower PQ and the distance XQ.

Solution

In ΔYRQ, we have $\tan 45° = \frac{QR}{YR}$



$\Rightarrow 1 = \frac{x}{YR}$

$\Rightarrow$ YR = x

$\Rightarrow$ XP = x [∵ YR=XP)

In ΔXPQ, we have $\tan 60° = \frac{PQ}{Px}$

$\Rightarrow \sqrt{3} = \frac{x + 40}{x}$

$\Rightarrow x\sqrt{3} = x + 40$

$\Rightarrow x\left(\sqrt{3} - 1\right) = 40$

$\Rightarrow x = \frac{40}{\sqrt{3} - 1}$

$\Rightarrow x = \frac{40}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1} = 20\left(\sqrt{3}+1\right) = 54.64$

So, height of the tower PQ = x+40 = 54.64+40 = 94.64 metres

In DXPQ, we have sin 60° = $\frac{PQ}{XQ}$

$\Rightarrow \frac{\sqrt{3}}{2} = \frac{94.64}{XQ}$

$\Rightarrow XQ = \frac{94.64 \times 2}{\sqrt{3}}$

$\Rightarrow XQ = \frac{94.64 \times 2 \times \sqrt{3}}{3}$ = 109.3 metres.

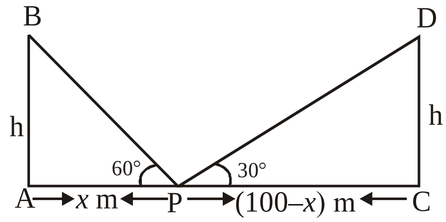
Example 4

Two pillars of equal height and on either side of a road, which is 100m wide. The angles of elevation of the top of the pillars are 60° and 30° at a point on the road between the pillars. Find the position of the point between the pillars and the height of each pillar.

Solution

Let AB and CD be two pillars, each of height h metres. Let P be a point on the road such that AP = x metres. Then, CP = (100–x) metres. It is given that $\angle$APB = 60° and $\angle$CPD = 30°

In Δ PAB, we have tan 60° = $\frac{AB}{AP}$



$\Rightarrow \sqrt{3} = \frac{h}{x}$ $\qquad \Rightarrow h = \sqrt{3}x$

In Δ PCD, we have tan 30° = $\frac{CD}{PC}$

$\Rightarrow \frac{1}{\sqrt{3}} = \frac{h}{100-x}$

$\Rightarrow h\sqrt{3} = 100 - x$ ....(ii)

Eliminating h between equation (i) and (ii), we get $3x - 100 - x \Rightarrow 4x = 100 \Rightarrow$ x =25

Substituting x - 25 in equation (i), we get

h = $25\sqrt{3} = 25 \times 1.732 = 43.3$

Thus, the required point is at a distance of 25 metres from the first pillar and 75 metres from the second pillar. The height of the pillars is 43.3 metres.

Example 5

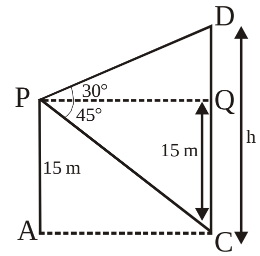
From a window 15 metres high above the ground in a street, the angles of elevation and depression of the top and the foot of another house on the opposite side of the shreet are 30° and 45° respectively show that the height of the opposite house is 23.66 metres (Take $\sqrt{3}=1.732$)

Solution

Let the window be P at a height of 15 metres above the ground and CD be the house on the opposite side of the street such that the angles of devation of the top D of house CD as seen from P is of 30° and the angle of depression of the foot C of house CD as seen from P is of 45°.

Let h metres of be the height of the house CD. we have,

QD=CD–CQ=CD–AP = (h–15) metres.



In Δ PQC, we have tan 45° = $\frac{QC}{PQ} \Rightarrow 1 \Rightarrow \frac{15}{PQ} = 15$ metres.

In Δ PQD, we have tan 30° = $\frac{QC}{PQ}$

$$\Rightarrow \frac{1}{\sqrt{3}} = \frac{h-15}{15} \Rightarrow h-15 = \frac{15}{\sqrt{3}} \Rightarrow h-15 = 5\sqrt{3}$$

$\Rightarrow$ h = 15 + 5 × 1.732 = 23.66 metress,

Hence, the height of the opposite house is 23.66 metres.

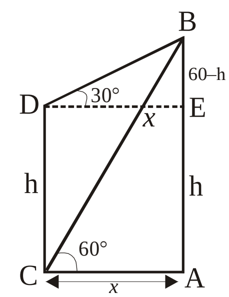
Example 6

From the top of a building 60 m high the angles of depression of the top and the bottom of a tower are observed to be 30° and 60°. Find the height of the tower.

Solution

Let AB be the building and CD be the tower. Let CD = h metres. Let DE be horizontal from D. It is given that the angles of depression of the top D and bottom C of the tower CD are 30° and 60° repectively.

$\therefore$ $\angle$EDB = 30° and $\angle$ACB = 60° Let AC = DE = x.



In ΔDEB, we have tan 30° = $\frac{BE}{DE}$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{60-h}{x}$$

$$\Rightarrow \quad x = (60-h)\sqrt{3}$$

In ΔCAB, we have tan 60° = $\frac{AB}{CA}$

$$\Rightarrow \sqrt{3} = \frac{60}{x}$$

$$\Rightarrow x = \frac{60}{\sqrt{3}}$$

From equation (i) and (ii), we have

$$(60-h)\sqrt{3} = \frac{60}{\sqrt{3}}$$

EXERCISE-I

1. The angle of depression of a boat from a bridge 25 mt. high is 45º. The horizontal distance of the boat from the bridge is.
2. A tower be 10 mt. high, the angle of depression of a point on the ground from the top of the tower is 30º. The distance of the point from base of the tower is.
3. The slope of a hill makes an angle of 60º, with the horizontal, if to reach the top one has to walk 500 mt., the height of the hill is.
4. If the angle of elevation of sun is 45º, the length of shadow of a tree 12 mt. high is.
5. The angle of elevation of a point in horizontal from the slope of a hill is 60º. If one has to walk 300 mt. to reach top, the distance of the point from base of the hill is.
6. A tower stands vertically on the ground. From a point on the ground, 20 mt. away from the foot of the tower, the angle of the elevation of the top of the tower is 60º. What is the height of the tower.
7. The angle of elevation of the top of a tower at a distance of 150 mt. from it's foot on a horizontal plane is found to be 60º. Find the height of the tower.
8. A tree is broken by the wind. The top struck the ground at an angle at 30º and at a distance of 30 mt. from the root. Find the whole height of the tree.
9. A boy 1.5 mt. tall, is 28.5 mt. away from a tower 30 mt. high. Determine the angle of elevation of the top of the tower from his eye.
10. Find the angle of elevation of the sun (sun's altitude), when the length of shadow of a vertical tower is equal to it's height?
11. From an aeroplane vertically above a straight horizontal road, the angle of depression of two stones at 1 km apart on opposite side of the aeroplane are observed be 'α' and 'β'. Show that the height of the aeroplane above the road is $\dfrac{\tan\alpha \tan\beta}{\tan\alpha + \tan\beta}$.
12. The angle of elevation of the top of a tower from point's at distance 'a' and 'b' units from the base on the same straight line with it are complementary. Prove that the height of the tower is $\sqrt{ab}$ units.
13. A vertical flagstaff stands on a horizontal plane. From a point 200 mt. from it's foot, the angle of elevation of it's top is found to be 30º. Find the height of the flagstaff.
14. A vertical tower stand on a horizontal plane and is surmounted by a vertical flagstaff of height 'h'. At a point on the plane, the angle of elevation of the bottom of the flagstaff is 'α' and that of the top of the flagstaff is 'β'. Prove that the height of the tower is $\dfrac{h\tan\alpha}{\tan\beta - \tan\alpha}$.

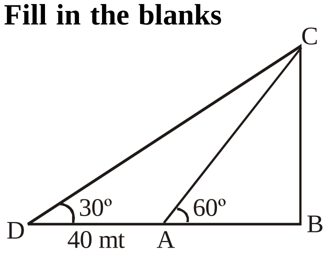
EXERCISE-II

1. The angle of elevation of the top of a tower from two points A and B at distance of p and q respectively from the base and in the same straight line with it, are complementory. Prove that the height of the tower is $\sqrt{pq}$.
2. From a building 60 mt. high the angle of depression of the top and bottom of lamp post are 30º and 60º respectively. Find the distance between lamp post and building. Also find the difference of height between building and lamp post.
3. As observed from the top of a light house, 100 mt. high above sea level, the angle of depression of a ship sailing between towards it, charges from 30º to 60º. Determine the distance travelled by the ship during the period of observation. (use $\sqrt{3}$ = 1.7323)
4. From an aeroplane vertically above a straight horizontal plane, the angle of depression of plane consecutive kilometre stone's on the opposite side, the aeroplane are found to be 'α' and 'β'. Show that the height of the aeroplane is $\frac{\tan\alpha \tan\beta}{\tan\alpha + \tan\beta}$
5. A tower is 50 mt. high, it's shadow is x mt. shorter when the sun altitude is 45º, than other it is 30º. Find the value of 'x' correct to nearest.
6. The angle of depression of the top and bottom of a tower as seen from the top of a 100 mt. high cliff are 30º and 60º respectively. Find the height of the tower.
7. The angle of elevation of a clouod from a point 200 mt. above the take is 30º and the angle of depression of the reflection of the cloud in the lake is 60º, find the height of the cloud.
8. The angle at elevation of an aeroplane from a point 'p' on the ground is 60º. After a flight at 15 second's the angle of elevation charges to 30º. If the aeroplane is flying at a constant height of $1500\sqrt{3}$ mt., find the speed of the aeroplane.
9. A bridge across a river makes an angle of 45º with the river bank. If the length of the bridge across the river is 150 mt, what is the width of the river.
10. The angle of elevation of the top of a tower from the point on the ground is 30º. After walking 30 mt. clouds the tower, the angle of elevation becomes 60º. What is the height of the tower.

EXERCISE-III

SECTION-A

- **Fill in the blanks**

1. 

AB = ____, CB = _______.

SECTION-B

- **Multiple choice question with one correct answers**

1. The angle of elevation of the top of a tower from the top and bottom of a building of height 'a' are 30º and 45º respectively. If the tower and the building stand at the same level, the height of the tower is.

(A) $a\sqrt{3}$ (B) $a\left(\sqrt{3}-1\right)$ (C) $a\dfrac{\left(3+\sqrt{3}\right)}{2}$ (D) $a\left(\sqrt{3}+1\right)$

2. From the top of a light house 60 mt. high with it's base at sea level, the angle of depression of a boat is 15º. The distance of the boat from the foot of the light house is

(A) $\left(\dfrac{\sqrt{3}-1}{\sqrt{3}+1}\right)60$ mt. (B) $\left(\dfrac{\sqrt{3}+1}{\sqrt{3}-1}\right)60$ mt. (C) $\left(\dfrac{\sqrt{3}-1}{\sqrt{3}+1}\right)$mt. (D) none of these

3. On level ground the angle of elevation of the top of the tower is 30º. On moving 20 mt. near, then angle of elevation is 60º. The height of the tower is.

(A) $20\sqrt{3}$ (B) $10\sqrt{3}$ (C) $10\left(\sqrt{3}-1\right)$ (D) none

4. A pole stand vertically inside a triangular park ABC. If the angle of elevation of the top of the pole from each corner of the park is same then in Δ ABC the foot of the pole is at

(A) cantriod (B) cirlum centre (C) incentre (D) ortho centre

5. A pole 25 mt. long stands on the top of a tower 225 mt. high. If 'θ' is the angle subtended by the pole at a point on the ground which is at a distance of 2.25 km from the foot of the tower, then tan θ is equal to

(A) 1/90 (B) 1/91 (C) 1/10 (D) 1/9

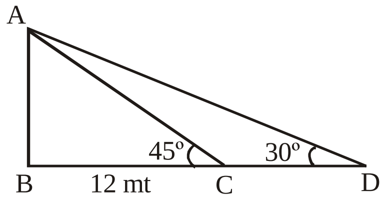
6. A tower stands vertically in a field. The field is in the shape of an equilateral triangle of side 100 mt. The tower subtend's angles of 30º, 60º at the vertices of a triangle. Find the height of the tower.

(A) $\frac{50}{\sqrt{3}}$ mt (B) $50\sqrt{3}$ mt. (C) $\frac{25}{\sqrt{3}}$ mt. (D) $25\sqrt{3}$ mt.

SECTION-C

- **Match the following (one to one)**

Column-I and **column-II** contains **four** entries each. Entries of column-I are to be matched with some entries of column-II. Only One entries of column-I may have the matching with the same entries of column-II and one entry of column-II Only one matching with entries of column-I



1. Column I | **Column II**

Column I	Column II
(A) AB	(P) $12\sqrt{2}$ mt.
(B) CD	(Q) 24 mt.
(C) AC	(R) 12 mt.
(D) AD	(S) 16.39 mt.

Answers

EXERCISE-I

1. 25 mt. 2. $10\sqrt{3}$ mt. 3. $250\sqrt{3}$ mt. 4. 12 mt. 5. 150 mt.

6. $20\sqrt{3}$ mt. 7. 259.80 mt. 8. 57.96 mt. 9. 45° 10. 45°

13. 115.5 mt.

EXERCISE-II

2. $20\sqrt{3}$ mt., 20 mt.
3. 115.47 mt.
5. 36.60 mt.
6. 66.67 mt.
7. 400 mt.
8. 720 km/h
9. $75\sqrt{2}$ mt.
10. $15\sqrt{3}$ mt.

EXERCISE-III

SECTION-A

1. AB = 20 mt., CB = 34.64 mt.

SECTION-B

1. (C)
2. (B)
3. (B)
4. (B)
5. (B)
6. (B)

SECTION-C

1. (A)-(R), (B)-(S), (C)-(P), (D)-(Q)